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Reduction of three-dimensional vector fields to the irreducible representation of inhomogeneous Lorentz group

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The reduction of wavefunction which transforms as a three dimensional space-vector for non-zero and zero mass systems, to the irreducible representation of the proper, orthochronous, inhomogeneous Lorentz group has been discussed. The results are used to derive the reduction of electromagnetic fields for nonzero and zero mass systems. For the second case the reduction and the second quantization of the fields have been discussed for linear as well as angular momentum basis. In the solution of Maxwell's equations for free-space the reduced expansion of electric and magnetic fields have been obtained as the components of an electromagnetic wave which is circularly polarised in opposite directions.

INTRODUCTION

The reduction of electromagnetic potential to the photon wavefunction in linear momentum basis according to the Wigner's classification (1939) of the relativistic particles corresponding to the irreducible representation of the proper, orthochronous, inhomogeneous Lorentz group, has been discussed by H. E. Moses (1966). This representation in linear momentum basis was transformed, (Moses 1967a) to the angular momentum basis using the general ways discussed by H. E. Moses (1965). The recipe of the discussion of H. E. Moses & J. S. Lomont (1967) enables one to reduce any unitary ray representation of the proper, orthochronous, inhomogeneous Lorentz group for both non-zero and zero mass systems (Moses 1967b, 1968), where for the former, one obtains the Foldy (1956)-Shirokov (1958) relations and for the latter, one is led to the Lomont-Moses realization (1964). Using these techniques we discussed the reduction of wavefunctions which transforms as complex antisymmetric tensor (Rajput 1969a, 1969b) and scalar (Rajput 1969c) fields. We extended these reductions for complex electromagnetic fields (Rajput 1969d) to discuss the group-theoretical nature of these fields.

In the present paper the reduction of wavefunction, which transforms as a three-dimensional space-vector for nonzero mass systems, to the irreducible representation of the inhomogeneous Lorentz group has been discussed. The results are used to derive the reduction of electromagnetic fields for nonzero and zero mass cases. We have also discussed the reduction and second quantization of these fields in angular momentum basis. In the solution of Maxwell's equation in free-space without source we obtain the reduced expansions of electric and magnetic fields of electromagnetic wave which is proved to be circularly polarized in opposite directions for $\lambda = +1$ and $\lambda = -1$.

TRANSFORMATION OF THE VECTORS.

In terms of components of antisymmetric tensors the components of electromagnetic fields are given by :

$$E_i\,(x,t) = F^{0i}\,(x,t)$$

$$H_i\,(x,t) = \epsilon_{ijk}\,F^{jk}, \qquad i, j, k = 1, 2, 3. \qquad \text{...(1)}$$

The components of antisymmetric tensors can be expressed in terms of three-dimensional space vector $\vec{A}\,(x)$ as follows :

$$F_{ij} = \frac{\partial}{\partial x_i}\,A_j - \frac{\partial}{\partial x_j}\,A_i \qquad \text{...(2)}$$

where,

$$\vec{A}\,(x) = \begin{bmatrix} A_1\,(x) \\ A_2\,(x) \\ A_3\,(x) \end{bmatrix} \qquad \text{...(3)}$$

Combining equations (1) and (2) we get :

$$\vec{E}\,(x,t) = -\frac{\partial}{\partial t}\,\vec{A}\,(x)$$

$$\vec{H}\,(x,t) = \text{Curl}\,\vec{A}\,(x). \qquad \text{...(4)}$$

Under the Lorentz transformations the vector $\vec{A}\,(x)$ transforms as follows :

$$A'\,(\vec{x}) = A\,[T\,(\vec{a})\,\vec{x}] = \text{Exp}\left[\sum_j a^j P_j\right] A\,(\vec{x})$$

$$A'\,(\vec{x}) = \text{Exp}\,(i\vec{\theta}.M)\;A\,[R\,(-\vec{\theta})\,\vec{x}] = \text{Exp}\,[i\vec{\theta}.K]\;A\,(\vec{x}) \qquad \text{...(5)}$$

$$A'\,(\vec{x}) = \text{Exp}\,(i\vec{\beta}.N)\;A\,[L\,(-\vec{\beta})\,\vec{x}] = \text{Exp}\,[i\vec{\beta}.Z]\;A\,(\vec{x})$$

where $T\,(\vec{a})$, $R\,(\vec{\theta})$ and $L\,(\vec{\beta})$ are Lorentz transformations corresponding to translation, rotation and pure Lorentz transformations, respectively. Matrices M and N are given by :

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{...(6)}$$

$$N_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

They satisfy the commutation rules of the infinitesimal generators of the proper, orthochronous, homogeneous Lorentz group. In the reduced form we can write M_i as :

$$M'_i = \begin{bmatrix} 0 & 0 \\ 0 & S' \end{bmatrix} \qquad \ldots(7)$$

where S', constitute the irreducible representation of the generators of the rotation group corresponding to the vector rotations such that :

$$\text{Exp}\ (i\vec{\theta}.S') = I + i\,(\vec{\theta}.S')\ \frac{\sin\theta}{\theta} + (\vec{\theta}.S')^2\ \frac{\cos\theta - 1}{\theta^2} = R(\vec{\theta})$$

Hence,

$$[\hat{R}\ (\vec{\theta})]_{\alpha\beta} = \delta_{\alpha\beta}\cos\theta - \frac{\theta_\alpha\theta_\beta}{\theta^2}\ (\cos\theta - 1) + \sum_\gamma \epsilon_{\alpha\beta\gamma}\ \theta_\gamma\ \frac{\sin\theta}{\theta} \qquad \ldots(8)$$

Similarly for $L\ (\vec{\beta})$ we have :

$$[L\ (\vec{\beta})]_{\alpha\beta} = [\text{Exp}\ (i\vec{\beta}.N)]_{\alpha\beta}$$

$$= \delta_{\alpha\beta}\cosh\beta - \frac{\beta_\alpha\beta_\beta}{\beta^2}\ (\cosh\beta - 1) + i\sum_\gamma \epsilon_{\alpha\beta\gamma}\beta_\gamma\ \frac{\sinh\beta}{\beta} \qquad \ldots(9)$$

The infinitesimal generators $P^0 = H$, P_j, K_j and Z_j of equation (5) constitute the proper, orthochronous, inhomogeneous Lorentz group and satisfy the following commutation rules :

$$[I^\alpha, P^\beta] = 0,\ [K_\alpha, K_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma}\ K_\gamma,$$

$$[K_\alpha, P_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma}\ P_\gamma,\ [Z_\alpha, Z_\beta] = -i\sum_\gamma \epsilon_{\alpha\beta\gamma}\ K_\gamma. \qquad \ldots(10)$$

Reduction of vectors for non-zero mass system.

Under the infinitesimal generators $\hat{P}^\alpha$, $\hat{K}_i$, $\hat{Z}_i$ of the unitary ray representation of Lorentz group, the complex function $f\ (\mu,\epsilon,\vec{p},\lambda)$ transforms in the following manner (Moses & Lomont 1967) :

$$\hat{P}_0 f\ (\mu,\epsilon,\vec{p},\lambda) = \hat{H} f\ (\mu,\epsilon,\vec{p},\lambda) = \epsilon\omega(\mu,p) f\ (\mu,\epsilon,\vec{p},\lambda),$$

$$\hat{P}_i f\ (\mu,\epsilon,\vec{p},\lambda) = p_i f\ (\mu,\epsilon,\vec{p},\lambda),$$

$$\hat{K}_i f\ (\mu,\epsilon,\vec{p},\lambda) = [-\sum_{jk} \epsilon_{ijk}\ p_j\ \frac{\partial}{\partial p_k} + S_i] f\ (\mu,\epsilon,\vec{p},\lambda), \qquad \ldots(11)$$

$$\hat{Z}_i f(\mu,\epsilon,\vec{p},\lambda) = \epsilon\left[i\omega(\mu,p)\frac{\partial}{\partial p_i} + \frac{1}{\omega(\mu,p)+\mu} \sum_{jk} \epsilon_{ijk}\, p_j S_k \right] f(\mu,\epsilon,\vec{p},\lambda).$$

where $\vec{p}$ has the the components $p_i (i = 1,2,,3)$, μ is the eigen value of the mass operator, $M = (H^2 - P^2)^{1/2}$, ϵ gives the sign of the energy ($\pm$ 1) and λ may have any value from 1 to $2s + 1$, where s is the spin corresponding to matrices S_i'. $\omega(\mu,p)$ is given by :

$$\omega(\mu,p) = [\mu^2 + p^2]^{1/2}.$$

These generators satisfy the commutation rules of the generators of Lorentz group. Hence for the required reduction we express $\hat{A}(x)$ in terms of $f(\mu,\epsilon,\vec{p},\lambda)$. For this we consider the function $f(\xi)$ as the representation of $\vec{A}(x)$ in the basis being characterised by the space of wavefunction in Hilbert space upon which the generators operate. ξ, collectively denote all the variables upon which the function depends in this basis. Then we have (Moses & Lomont 1967),

$$f(\xi) = A(\vec{x}) = \sum_{\epsilon}\int d\mu \sum_{\lambda}\int \frac{dp}{\omega(\mu,p)+\mu} \times \langle \xi \mid \mu,\epsilon,p,\lambda \rangle f(\mu,\epsilon,p,\lambda) \quad ...(12)$$

where,

$$\langle \xi \mid \mu,\epsilon,p,\lambda\rangle = \langle x,t,\gamma \mid \mu,\epsilon,p,\lambda\rangle = \mathrm{Exp}\,[-i\vec{\beta}.Z]\, g(\xi,\mu,\epsilon,\lambda). \quad ...(13)$$

$g(\xi,\mu,\epsilon,\lambda)$ satisfies following equations :

$$P_i\, g(x,t,\gamma;\mu,\epsilon,\lambda) = 0$$

$$H\, g(x,t,\gamma;\mu,\epsilon,\lambda) = \epsilon\mu g(x,t,\gamma;\mu,\epsilon,\lambda)$$

Using the values of operators P_i, H we have Rajput (1969a) :

$$\frac{\partial}{\partial x_i} g(x,t,\gamma;\mu,\epsilon,\lambda) = 0$$

$$i\frac{\partial}{\partial t} g(x,t,\gamma;\mu,\epsilon,\lambda) = \epsilon\mu g(x,t,\gamma;\mu,\epsilon,\lambda) \quad . (14)$$

solution of which may be written as :

$$g(x,t,\gamma;\mu,\epsilon,\lambda) = C(\gamma \mid \mu,\epsilon,\lambda)\exp(-i\epsilon\mu t) \quad ...(15)$$

where $C(\gamma \mid \mu,\epsilon,\lambda)$ is the constant of integration in which for convenience λ is chosen to go through the same range of values as γ,

so, $$C(\gamma \mid \mu,\epsilon,\lambda) = C(\mu,\epsilon)\delta_{\gamma,\lambda} \quad .. (16)$$

Using equations (5) and (16) in (15) we have :

$$<\xi \mid \mu,\epsilon,p,\lambda> = C(\mu,\epsilon) \exp\,[i\{\vec{p}.\vec{x} - \epsilon\omega(\mu,p)t\}]$$
$$\times \exp\,[-i\,\vec{\beta}.N]_{\gamma\lambda} \qquad \ldots(17)$$

Now for each set of the elements along the principal diagonal of M_i given by equation (7) we define the column vector $\chi^{\alpha}(\mu,\epsilon,p,\lambda)$ with components :

$$\chi^{\alpha}(\gamma \mid \mu,\epsilon,p,\lambda) = \{\exp\,(-i\vec{\beta}.N)\}_{\gamma,\lambda} \qquad \ldots(18)$$

where α for null-matrix on principal diagonal of M_i is zero and hence the variable λ takes only one value and it need not be indicated, while for S', (the second elements on the principal diagonal of M_i) α is 1 for which $\lambda = 1,2,3$.

Using equations (9) and (18) we have :

$$\chi^0(0 \mid \mu,\epsilon,p) = \frac{\omega(\mu,p)}{\mu},$$

$$\chi^0(\gamma \mid \mu,\epsilon,p) = \frac{\epsilon}{\mu}\,p_{\gamma},$$

$$\chi^1(0 \mid \mu,\epsilon,p,\lambda) = \left(\frac{\epsilon}{\mu}\right)p_{\lambda}$$

$$\chi^1(\gamma \mid \mu,\epsilon,p,\lambda) = \frac{1}{\mu}\left[\mu\delta_{\gamma\lambda} + \frac{p_{\gamma}p_{\lambda}}{\omega(\mu,p)+\mu}\right], \qquad \lambda,\gamma = 1,2,3. \ldots(19)$$

The function $f\,(\mu,\epsilon,\vec{p}\,\lambda)$ in equation (12) represents $f^0(\mu,\vec{p})$ for $\gamma = 0$ and $f\,(\mu,\vec{p})$ for $\gamma = 1$.

Using equation (17) and (19) the equation (12) reduces to :

$$A(\vec{x}) = \sum_{\epsilon=\pm 1} \int dM^0(\mu,\epsilon) \int \frac{d\vec{p}}{\omega(\mu,p)}\,\vec{p}f^0\,(\mu,\epsilon,\vec{p})$$
$$\times \exp\,[i\{\vec{p}.\vec{x} - \epsilon\omega(\mu,\vec{p})t\}]$$
$$+ \sum_{\epsilon=\pm 1} \int dM^1(\mu,\epsilon) \int \frac{d\vec{p}}{\omega(\mu,p)}\left[\mu f\,(\epsilon,\mu,\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{f}\,(\epsilon,\mu,\vec{p})\}]}{\omega(\mu,\vec{p})+\mu}\right.$$
$$\times \exp\,[i\{\vec{p}.\vec{x} - \epsilon\omega(\mu,\vec{p})t\}] \qquad \ldots(20)$$

where $dM(\mu,\epsilon) = C(\mu,\epsilon)d\mu$ (measure function).

Equation (20) may be written as :

$$A(\vec{x}) = \int dM^0(\mu) \int \frac{d\vec{p}}{\omega(\mu,p)} \vec{p} f^0(\mu,\vec{p}) \exp [i\{\vec{p}.\vec{x} - \omega(\mu,p)\,t\}]$$

$$+ \int dN^0(\mu) \int \frac{d\vec{p}}{\omega(\mu,p)} \vec{p} h^{0*}(\mu,\vec{p}) \exp [-i\{\vec{p}.\vec{x} - \omega(\mu,p)t\}]$$

$$+ \int dM^1(\mu) \int \frac{d\vec{p}}{\omega(\mu,p)} [\mu \vec{f}(\mu,\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{f}(\mu,\vec{p})\}}{\omega(\mu,p)+\mu}] \exp [i\{\vec{p}.\vec{x} - \omega(\mu,p)t\}]$$

$$+ \int dN^1(\mu) \int \frac{d\vec{p}}{\omega(\mu,p)} [\mu h^*(\mu,\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{h}^*(\mu,p)\}]}{\omega(\mu,p)+\mu}$$

$$\times \exp [-i\{\vec{p}.\vec{x} - \omega(\mu,p)t\}] \quad \ldots(21)$$

where,

$$M^\gamma(\mu) = M^\gamma(\mu, +1),\ N^\gamma(\mu) = M^\gamma(\mu, -1)$$

$$f^\gamma(\mu,\vec{p}) = f^\gamma(\mu, +1,\vec{p})$$

and,

$$h^\gamma(\mu,\vec{p}) = f^{*\gamma}(\mu, -1, -\vec{p})$$

Applying reality condition, Lorentz condition and the requirement that $A(\vec{x})$ satisfies wave equation, the expression (21) reduces to :

$$A(\vec{x}) = C \int \frac{d\vec{p}}{\omega(p)} \exp [i\{\vec{p}.\vec{x} - \omega(p)t\}] \times [m\vec{f}(p) \frac{\vec{p}\{\vec{p}.\vec{f}(p)\}]}{\omega(p)+m}$$

$$+ D \int \frac{d\vec{p}}{\omega(p)} \exp [-i\{\vec{p}.\vec{x} + \vec{\omega(p)}t\}]$$

$$\times [mh^*(\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{h}(p)\}]}{\omega(p)+m} \quad \ldots(22)$$

where C and D are positive real constants and,

$$\vec{f}(p) = f(m,\vec{p}),\ \vec{h}(p) = h(m,\vec{p})$$

$$\omega(p) = \omega(m,p)$$

Using canonical formalism (Rajput 1969a) the constants C and D are calculated to have the following values :

$$C = D = (2)^{-1/2}\,(2\pi)^{-3/2}.$$

Hence,

$$A(\vec{x}) = \frac{1}{4\pi^{3/2}} \int \frac{d\vec{p}}{\omega(\vec{p})} [mf(\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{f}(\vec{p})\}]}{\omega(p)+\mu}$$

$$\times \exp\left[i\{\vec{p}.\vec{x} - \omega(p)t\}\right]$$

$$+ \frac{1}{4\pi^{3/2}} \int \frac{d\vec{p}}{\omega(p)} \; [mh^*(\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{h}^*(\vec{p})\}]}{\omega(p)+\mu}$$

$$\times \exp\left[-i\{\vec{p}.x - \omega(p)t\}\right] \qquad \text{..(23)}$$

The inner product of two vectors $A(\vec{x})$ and $A_1(\vec{x})$ in configuration space is defined as :

$$(A_1, A) = \int \frac{d\vec{p}}{\omega(p)} \; f_1^*(\vec{p}).\vec{f}(\vec{p}) + \int \frac{d\vec{p}}{\omega(p)} \; h^*(\vec{p}).\vec{h}_1(\vec{p}).$$

Using equations (23) and (4) we can now reduce the electromagnetic fields to the irreducible representation of inhomogenous Lorentz group for non-zero mass system. We get :

$$\vec{E} = \frac{i}{4\pi^{3/2}} \sum_{\epsilon=\pm 1} \epsilon \int d\vec{p}[mf(\epsilon,\vec{p}) + \frac{\vec{p}\{\vec{p}.\vec{f}(\epsilon,\vec{p})\}]}{\omega(p)+\mu} \times \exp[i\{\vec{p}.\dot{x} - \epsilon\omega(p)t\}] \qquad \text{.. (24)}$$

and,

$$\vec{H} = \frac{i}{4\pi^{3/2}} \sum_{\epsilon=\pm 1} \int m \frac{d\vec{p}}{\omega(p)} \; \{\vec{p} \times f(\epsilon,\vec{p})\} \times \exp\left[i\{\vec{p}.\vec{x} - \epsilon\omega(p)t\}\right]$$

$$= \frac{i}{4\pi^{3/2}} \int m \frac{d\vec{p}}{\omega(p)} \Big[\{\vec{p} \times \vec{f}(p)\} \exp\left[i\{\vec{p}.\vec{x} - \omega(p)t\}\right]$$

$$- \{\vec{p} \times \vec{h}^*(p)\} \exp\left[-i\{\vec{p}.\vec{x} - \omega(p)t\}\right]\Big] \qquad \text{...(25)}$$

Using Maxwell's equation :

$$\text{Curl } H = 4\pi j + \frac{\partial \epsilon}{\partial t}, \quad \text{and} \quad \text{div } E = 4\pi\rho \qquad \text{...(26)}$$

we may find ρ and j for the required field. The second of equation (26) gives :

$$4\pi\rho = -\frac{1}{4\pi^{3/2}} \int_{-\alpha}^{\alpha} d\vec{p}\omega(\vec{p}) \; [\vec{p}.\vec{f}(p) \exp i\{\vec{p}.\vec{x} - \omega(\vec{p})\vec{t}\}$$

$$+ \vec{p}.\vec{h}^*(p) \exp -i\{\vec{p}.\vec{x} - \omega(p)t\}] \qquad \text{...(27)}$$

Current density $\vec{j}$ can be reduced into similar expression to (23) in terms of $j\,(\epsilon, \vec{p})$ as follows :

$$\vec{j}(\vec{x}) = \frac{1}{4\pi^{3/2}} \sum_{\epsilon \pm 1} \int \frac{d\vec{p}}{\omega(p)} \left[mj(\epsilon,\vec{p}) + \frac{\vec{p}\{\vec{p}.j(\epsilon,\vec{p})\}}{\omega(p)+m} \right]$$

$$\times \exp\,[i\{\vec{p}.\vec{x} - \epsilon\omega(p)t\}]$$

Then first of equation (26) gives the following result :

$$4\pi \left[mj(\epsilon,\vec{p}) + \frac{\vec{p}\{\vec{p}.j(\epsilon,\vec{p})\}]}{\omega(p) + m} \right] = m[p^2 f(\epsilon,\vec{p}) - \vec{p}\{\vec{p}.f(\epsilon,\vec{p})\}]$$

$$- \omega^2\,(p) \left[mf(\epsilon,\vec{p}) - \frac{\vec{p}\{\vec{p}.f(\epsilon,\vec{p})\}}{\omega(p) + m} \right] \quad ..\,(28)$$

Second quantization of electromagnetic fields for non-zero mass system

To second quantize the eletromagnetic field in the basis the components $f\,(\vec{p})$ and $h(\vec{p})$ are considered as destruction and creation operators, respectively. Assuming Bose-statistics we require that all the commutators vanish except the following :

$$[f(\vec{p},\lambda), f^*(\vec{p}', \lambda')] = \omega(p)\,\delta(\vec{p} - \vec{p}')\,\delta_{\lambda,\lambda'}$$

$$[h(\vec{p},\lambda),\ h^*(\vec{p}',\lambda')] = \omega(p)\,\delta(\vec{p} - \vec{p}')\,\delta_{\lambda,\lambda'} \quad ...(29)$$

We thus regard the components of eletric and magnetic fields as operators. If $\hat{A}$ is any of the operators $\hat{H}$, $\hat{P}_i$, $\hat{K}_i$, $\hat{Z}_i$ and if $\hat{A}\,f\,(\vec{p})$, $\hat{A}h\,(\vec{p})$ are the operators formed when $\hat{A}$ acts on $\vec{p}$ and λ, as though $f(\vec{p})$ and $h(\vec{p})$ were representatives instead of destruction operators, then for each operator $\hat{A}$ we define a second quantized operator :

$$[A] = \sum_{\lambda} \int \frac{d\vec{p}}{\omega(p)}\, f^*(\vec{p},\lambda)\hat{A}f(\vec{p},\lambda) + \sum_{\lambda} \int \frac{d\vec{p}}{\omega(p)}\, h^*(\vec{p},\lambda)\hat{A}\,h(\vec{p},\lambda) \quad ..\,(30)$$

The operator E transforms under translation, rotation and pure Lorentz transformation as follows ;

$$E'(x) = \text{Exp}\,\{-i\sum_j a^j\,[\,P_j\,]\}\; E(x)\; \exp\,\{i\sum_j a^j\,[\,P_j\,]\}$$

$$E'(x) = \text{Exp}\,\{-i\,\vec{\theta}.\,[K]\}\; E(x)\; \exp\,\{i\,\vec{\theta}.\,[K]\}$$

$$E'(x) = \text{Exp}\,\{-i\,\vec{\beta}.\,[Z]\}\; E(x)\; \exp\,\{i\,\vec{\beta}.\,[Z]\}$$

The similar transformation can be derived for magnetic field operator $H(x)$

Reduction for zero mass system

The mass-less representation of the infinitesimal generators of inhomogenous Lorentz group is given in terms of the representation of infinitesimal generators (T_1, T_2, J) of two dimensional Euclidean group,

$$\begin{aligned} T_1 &= -M_2 - N_1 \\ T_2 &= M_1 - N_2 \\ J &= M_3 \end{aligned} \qquad \ldots(32)$$

For a real function $f(\lambda)$ of real variable λ (the eigenvalues of matrix M_3) the realization of the infinitesimal generators of inhomogeneous Lorentz group discussed by Lomont & Moses (1967) can be taken as :

$$\hat{P}_0 f(\vec{p}) = Hf(\vec{p}) = \epsilon p f(\vec{p})$$

$$P_i f(\vec{p}) = p_i f(\vec{p})$$

$$\hat{K}_1 f(\vec{p}) = \left[L_1 + \frac{p_1}{p+p_3} J\right] f(\vec{p})$$

$$\hat{K}_2 f(\vec{p}) = \left[L_2 + \frac{p_2}{p+p_3} J\right] f(\vec{p})$$

$$\hat{K}_3 f(\vec{p}) = [L_3 + J]\, f(\vec{p})$$

$$\hat{Z}_1 f(\vec{p}) = \epsilon \left\{ ip\frac{\partial}{\partial p_1} + \frac{p_2}{p+p_3} J + \left[\frac{p_1^2}{p^2(p+p_3)} - \frac{1}{p}\right] T_1 + \frac{p_1 p_2 T_2}{p^2(p+p_3)} \right\} f(\vec{p})$$

$$\hat{Z}_2 f(\vec{p}) = \epsilon \left\{ ip\frac{\partial}{\partial p_2} - \frac{p_1 J}{p+p_3} + \frac{p_1 p_2}{p^2(p+p_3)} T_1 + \left[\frac{p_2^2}{p^2(p+p_3)} - \frac{1}{p}\right] T_2 \right\} f(\vec{p})$$

$$\hat{Z}_3 f(\vec{p}) = \epsilon \left\{ ip\frac{\partial}{\partial p_3} + \frac{1}{p^2}\left[\, p_1 T_1 + p_2 T_2 \,\right] \right\} f(\vec{p}) \qquad \ldots(33)$$

where

$$L_i f(\vec{p}) = -i \sum_{jk} \epsilon_{ijk}\, p_j \frac{\partial}{\partial p_k}$$

The required reduction requires the expression of the vector $A(\vec{x})$ in terms of $f(\vec{p})$. The matrix M_3 defined in equation (6) is Hermitian which can be diagonalized by a unitary matrix U :

$$U^{-1} M_3 U = d. \qquad \text{(diagonal matrix)}$$

λ, the eigen values of M_3 are give by :

$$| M_3 - \lambda I | = 0$$

which gives $\lambda = 1,0,0,-1$.

So,

$$d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Hence,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(2)^{-1/2} & 0 & (2)^{-1/2} \\ 0 & -i(2)^{-1/2} & 0 & -i(2)^{-1/2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{.. (34)}$$

Now we define a column vector $\chi(\epsilon,p,\lambda)$ components of which are given as :

$$\chi(\gamma \mid \epsilon,p,\lambda) = [\exp\, (i\vec{\omega}.M) \exp\, (i\nu N_3\,]_{\gamma\lambda} \qquad (35)$$

where the correspondence of $\vec{p}$ with $\vec{\omega}$ and ν is given by the following expressions :

$$p = \operatorname{Exp}\,(\epsilon\nu)$$

$$p_1 = -p\left(\frac{\sin\omega}{\omega}\right)\omega_2$$

$$p_2 = p\left(\frac{\sin\omega}{\omega}\right)\omega_1$$

$$p_3 = \cos\omega$$

$$\omega_3 = 0 \qquad \text{...(36)}$$

Using equation (8) and (9) in equation (35) we have :

$$\chi(\epsilon,\vec{p},0) = \frac{1}{p}\begin{bmatrix} 0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\chi(\epsilon,\vec{p},\lambda) = \frac{\lambda}{(2)^{1/2}}\begin{bmatrix} \frac{p_1(p_1+i\lambda p_2)}{p(p+p_3)} - 1 \\ \frac{p_2(p_1+i\lambda p_2)}{p(p+p_3)} - i\lambda \\ \frac{p_1+i\lambda p_2}{p} \end{bmatrix}$$

$$\frac{\lambda}{(2)^{1/2}}\begin{bmatrix} 0 \\ \vec{\sigma}(p) \end{bmatrix} \quad \ldots(37)$$

Now equation (12) for this case reduces to :

$$f(\xi) = A(\vec{x}) = \sum_{\epsilon\pm 1} \sum_{\lambda=0\pm 1} \int \frac{d\vec{p}}{p} <\xi \mid \epsilon,p,\lambda> f(\epsilon,p,\lambda) \quad .(38)$$

where (Rajput 1969b),

$$<\xi \mid \epsilon,p,\lambda> = <x,t,\gamma \mid \epsilon,p,\lambda>$$
$$= \mathrm{Exp}\,[i(x_3 - \epsilon t)]\,\chi(\gamma \mid \epsilon,p,\lambda)$$

Hence equation (38) reduces to :

$$A(\vec{x}) = \sum_{\epsilon}\sum_{\lambda} C(\epsilon,\lambda) \int \frac{d\vec{p}}{p}\, \chi(\epsilon,p,\lambda) f(\epsilon,p,\lambda) \;\exp\;[i\{\vec{p}.\vec{x} - \epsilon pt\}]$$

$$= \sum_{\epsilon} [C(\epsilon,0) \int \frac{d\vec{p}}{p}\, \chi(\epsilon,p,0) f(\epsilon,p,0) \;\exp\;\{i(\vec{p}.\vec{x} - \epsilon pt)\}$$

$$+ \sum_{\lambda=\pm 1} C(\epsilon,\lambda) \int \frac{d\vec{p}}{p}\, \sigma(\epsilon,p,\lambda) f(\epsilon,p,\lambda) \;\exp\;\{i(\vec{p}.\vec{x} - pt)\}] \quad ..(39)$$

Now let,

$$h(\epsilon,\vec{p},\lambda) = \mathrm{Exp}\,(-2i\lambda\phi) f^*(-\epsilon, -\vec{p}, \lambda)$$

where, $\tan\phi = p_2 \mid p_1$

$C(0) = C(+1,0),\quad D(0) = C(-1,0)$

$C(\lambda) = C(+1,\lambda),\quad D(\lambda) = D(-1,\lambda)$

$f(\vec{p},\lambda) = f(+1,\vec{p},\lambda)$

$\chi(\vec{p},0) = \chi(+1,\vec{p},0)$

$\eta(\vec{p},0) = \chi(-1,\vec{p},0) = -\chi(\vec{p},0)$

$$\chi(\vec{p},\lambda) = \chi(+1,\vec{p},\lambda) = \frac{\lambda}{(2)^{1/2}}\begin{bmatrix} 0 \\ \sigma(p) \end{bmatrix}$$

$$\eta(\vec{p},\lambda) = \chi(-1,\vec{p},\lambda) = -\chi^{*}(\vec{p},\lambda)$$

Then,

$$\vec{A}(x) = C(0)\int \frac{d\vec{p}}{p}\, \chi(\vec{p},0) f(\vec{p},0) \exp\{i(\vec{p}.x - pt)\}$$

$$- D(0)\int \frac{d\vec{p}}{d}\, \chi(\vec{p}.0)\, h^{*}(\vec{p},0) \exp\{-i(\vec{p}.\vec{x} - pt)\}$$

$$+ \sum_{\lambda}\Big[C(\lambda)\int \frac{d\vec{p}}{p}\, \sigma(\vec{p},\lambda) f(\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - pt)\}$$

$$-D(\lambda)\int \frac{d\vec{p}}{p}\, \sigma^{*}(\vec{p},\lambda)\, h^{*}(\vec{p},\lambda) \exp\{-i(\vec{p}.\vec{x} - pt\}\Big] \quad ...(40)$$

For Γ any of the infinitesimal generators P_i, H, J_i, $\Gamma A(x)$ has the same expression (40) on replacing $f(\epsilon,p,\lambda)$ by $\hat{\Gamma} f(\epsilon,p,\lambda)$.

But for $\hat{\Gamma}$ one of Z_i, $\hat{\Gamma}$ will not be Hermitian and $\hat{\Gamma}A(x)$ consists of two parts one of which corresponds to a true physical change called change of gauge :

$$Z_i\, A(x) = \sum_{\epsilon}\sum_{\lambda} C(\epsilon,\lambda)\int \frac{d\vec{p}}{p}\, \chi(\epsilon,\vec{p},\lambda)\, \hat{Z}^i f(\epsilon,\vec{p},\lambda)$$

$$\times \exp\{i(\vec{p}.\vec{x} - \epsilon pt)\} + G_i(x) \quad ...(41)$$

where $G_i(x)$ is the gauge change given by (Rajput 1969b) :

$$G_j(x) = i\Big[(m\times\nabla)_j + \frac{\partial}{\partial t} N_j\Big]\phi_1(x) + i(N.\nabla)_i \nabla_j \frac{\partial}{\partial t}\phi_3(x).$$

$$= -\nabla\, \phi_{1,j}(x) + \vec{e}_j[\nabla.\phi_1(x)] - i\frac{\partial}{\partial t}[e_j \times \phi_1(x)]$$

$$- i\frac{\partial^2}{\partial x_j \partial t}[\nabla \times \phi_2(x)]$$

where,

$$\phi_n(x) = \sum_{\epsilon}\sum_{\lambda} C(\epsilon,\lambda)\int \frac{d\vec{p}}{p^{n+1}}\, \chi(\epsilon,\vec{p},\lambda) f(\epsilon,\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - \epsilon pt)\} \quad ...(42)$$

$\vec{e}_j$ is the unit vector in the direction of j-th space axis and $\phi_{1,j}$ denotes the j-th component of the vector $\phi_1(x)$

If $A(x)$ is a real vector,

$$A(x) = \overset{*}{A}(x)$$

then, $$C(0)\, f(\vec{p},0) = -D(0)\, h(\vec{p},0)$$

and, $$C(\lambda)\, f(\vec{p},\lambda) = -D(\lambda) h(\vec{p},\lambda).$$

Hence the real vector is expressed as :

$$A(x) = C(0) \int \frac{d\vec{p}}{p} \chi(\vec{p},0)[\, f(\vec{p},0) \exp \{i(\vec{p}, \vec{x} - pt)\} + f^*(p,0) \exp \{-i(\vec{p}.\vec{x} - pt)\}]$$
$$+ \sum_{\lambda=+1} C(\lambda) \int \frac{d\vec{p}}{p} [\sigma(\vec{p},\lambda)\, f(\vec{p},\lambda) \exp \{i(\vec{p}, \vec{x} - pt)\} + \sigma^*(\vec{p},\lambda)\, f^*(\vec{p},\lambda) \exp \{-i\, (\vec{p}.\vec{x})\} \quad \ldots(43)$$

Using equation (4) and (43) we can reduce the electromagnetic fields to the irreducible representation of inhomogeneous Lorentz group for zero mass system. Thus we get :

$$\vec{E}(\vec{x}) = iC(0) \int d\vec{p} \chi(\vec{p},0)[\, f(\vec{p}.0) \exp \{i(\vec{p}.\vec{x} - pt)\} - f^*(\vec{p},0) \exp \{-i(\vec{p}.\vec{x} - pt)\}]$$
$$+ i \sum_{\lambda=\pm1} C(\lambda) \int d\vec{p}[\sigma(\vec{p},\lambda)\, f(\vec{p},\lambda) \exp \{i(\vec{p}.\vec{x} - pt)\} - \sigma^*(\vec{p},\lambda)\, f^*(\vec{p},\lambda) \exp \{-i(\vec{p}.\vec{x} - pt)\}] \quad \ldots(44)$$

$$H(\vec{x}) = \sum_{\lambda=\pm1} \lambda C(\lambda) \int d\vec{p}[\sigma(\vec{p},\lambda)\, f(\vec{p},\lambda) \exp \{i(\vec{p}.\vec{x} - pt)\} + \sigma^*(\vec{p},\lambda)\, f^*(\vec{p},\lambda) \exp \{-i(\vec{p}.\vec{x} - pt)\}] \quad \ldots(45)$$

We can write equations (44) and (45) as :

$$E(x) = E^L + E^T$$
$$H(x) = H^L + H^T$$

where E^L and E^T are longitudinal and transverse parts of electric field given by :

$$E^L = iC(0) \int d\vec{p}\, \chi\, (\vec{p},0)[\, f(\vec{p},0) \exp \{i(\vec{p}.\vec{x} - pt)\} + f^*(\vec{p},0) \exp \{-i\vec{p}.\vec{x} - pt)\}] \quad \ldots(46)$$

$$E^T = \sum_{\lambda\pm1} iC(\lambda) \int d\vec{p}[\sigma(\vec{p},\lambda)\, f(\vec{p},\lambda) \exp \{i(\vec{p}.\vec{x} - pt)\} - \sigma(\vec{p},\lambda)\, f^*[\vec{p},\lambda) \exp \{-i(\vec{p}.\vec{x} - pt\} \quad \ldots(47)$$

Similarly the transverse part of magnetic fields is given by equation (45) :

$H^T = H(x)$, while, $H^L = 0$.

Thus the longitudinal part of magnetic field in electro-magnetic wave is always zero.

In the free space Maxwell's equations lead to :

$$f(\vec{p},0) = 0)$$

Hence equation (44) reduces to :

$$\vec{E}(x) = i \sum_{\lambda=+1} C(\lambda) \int d\vec{p}[\sigma(\vec{p},\lambda)f(\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - pt)\}$$
$$-\sigma^*(\vec{p},\lambda) f^*(\vec{p},\lambda) \exp\{-i(\vec{p}.\vec{x} - pt)\}] \quad ..(48)$$

It is clear at this stage that $\phi_n(x)$ given by equation (42) satisfies Maxwell's equation when $\vec{E}$ and $\vec{H}$ do. Hence the gauge changes are identically zero. So $\vec{E}$ and $\vec{H}$ transform without the necessity of introducing the gauge changes.

We choose the value of the constant such that the usual canonical formalism in terms of Hamiltonian density agrees with the particle interpretation. Hamiltonian density of the electromagnetic fields is defined as :

$$H(x) = \frac{1}{8\pi}(E^2 + H^2) \quad ...(49)$$

The energy of the field is :

$$T = \int_{-\infty}^{\infty} H(x)dx$$
$$= (8\pi)^{-1} \int_{-\infty}^{\infty} (E^2 + H^2)dx$$

while the expectation energy is given by :

$$T_f = \int_{-\infty}^{\infty} \frac{dp}{p} f^*(\vec{p}, +1)\vec{p}f(\vec{p}, +1)$$
$$\int_{-\infty}^{\infty} dp \mid f(p, +1) \mid^2 \quad ...(50)$$

Equations (49) and (50) lead to :

$$C(\lambda) = (8\pi^2)^{-1/2}$$

Hence,

$$E(x) = i(8\pi^2)^{1/2} \sum_{\lambda} \int d\vec{p}[\sigma(\vec{p},\lambda) f(\vec{p},\lambda) \exp\{i(\vec{p}.x - pt)\}$$
$$- \sigma^*(\vec{p},\lambda) f^*(\vec{p},\lambda) \exp\{-i(\vec{p}.\vec{x} - pt)\}] \quad .. (51)$$

$$H(x) = (8\pi^3)^{-1/2} \sum_{\lambda=\pm 1} \lambda \int d\vec{p}[\sigma(\vec{p},\lambda) f(\vec{p},\lambda) \exp\{i(\vec{p}\cdot\vec{x} - pt)\} + \sigma^*(\vec{p},\lambda) f^*(\vec{p},\lambda) \exp\{-i(\vec{p}.\vec{x} - pt)\} \quad ...(52)$$

Equations (51) and (52) can be written as follows :

$$E(x) = E_1(x,t) + E_1^*(x,t). \quad ...(51a)$$

where,

$$E_1(x,t) = \frac{i}{(8\pi^2)^{1/2}} \sum_{\lambda=\pm 1} \int d\vec{p} f(\vec{p},\lambda)\sigma(\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - pt)\}$$

and

$$H(x) = H_1(x,t) + H_1^*(x,t) \quad ...(52a)$$

where,

$$H_1(x,t) = \frac{1}{(8\pi^2)^{1/2}} \sum_{\lambda=\pm 1} \lambda \int d\vec{p} f(\vec{p},\lambda)\sigma(\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - pt)\}$$

In equation (51) if we take :

$$f(\vec{p},\lambda) = \delta_{\lambda,1}\delta(p_1 - k)\delta(p_2)\delta(p_3) \quad ..(53)$$

then the function $f(\vec{p},\lambda)$ represents in the limiting case a wave function of photon of momentum k moving in the positive x-direction for $\lambda = +1$.

For this value of $f(\vec{p},\lambda)$ we have :

$$\sigma(\vec{p},\lambda) f(\vec{p},\lambda) = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

and hence equation (51) gives :

$$E_1 = 0$$

$$E_2 \sim \cos k(t - x)$$

$$E_3 \sim \sin k(t - x) \quad ...(54)$$

The wave represented by equation (54) is circularly polarized in the positive x-direction. Similarly we can show that $\lambda = -1$ gives circular polarization in the opposite direction. The x-component of the Poynting vector is given by :

$$\frac{1}{4\pi}[\vec{E} \times \vec{H}]_x = \frac{1}{8\pi^3} \quad ...(55)$$

while the other components are zero,

To second quantize the electromagnetic fields given by equations (55) and (52) the function $f(\vec{p},\lambda)$ and its complex conjugate are considered as destruction and creation operators, respectively. Using Bose-statistics we get the following commutation relations for these operators :

$$[f(\vec{p},\lambda), f(\vec{p}\,',\lambda')] = [f^*(\vec{p},\lambda), f^*(\vec{p}\,',\lambda')] = 0$$

$$[f(\vec{p},\lambda), f^*(\vec{p}\,',\lambda')] = p\delta_{\lambda,\lambda'}\delta(\vec{p}-\vec{p}\,') \quad ..(56)$$

For these operators electric and magnetic fields are also considered as operators which satisfy usual commutation rules. For every infinitesimal generator $\hat{A}$ we can define a second quantized operator [A] by :

$$[\mathrm{A}] = \sum_{\lambda} \left[\int \frac{d\vec{p}}{p} f^*(\vec{p},\lambda)\, \hat{A}\, f\,(\vec{p},\lambda) \right] \quad ..(57)$$

Under the Lorentz transformations the electric and magnetic fields transform according to the the equations (31) where second quantized operator is given by equation (57)

REDUCTION OF ELECTROMAGNETIC FIELDS IN ANGULAR MOMENTUM BASIS

In angular momentum basis wave function depends on the energy, the total angular momentum quantum number k, the quantum number m for J_z (z-component of angular momentum) and the helicity λ. In this case let this function be $F(p,k,m,\lambda)$ then we have (Moses 1965, Lomont & Moses 1967)

$$f(\vec{p},\lambda) = \frac{1}{p} \sum_{k=1}^{\infty} \sum_{m=-k}^{k} \mathrm{Exp}\left\{ i\pi\left(\lambda - \frac{m}{2}\right)\right\} Y_k^{m\lambda}(\theta,\phi) F(p,k,m,\lambda) \quad ...(58)$$

where $Y_k^{m\lambda}(\theta,\phi)$ are generalised spherical harmonics defined as :

$$Y_k^{m\lambda}(\theta,\phi) = (-1)^{m-\lambda}\,(\tfrac{1}{2})^{m+1} \left[\frac{(2k+1)}{\pi}\right]^{1/2} \left[\frac{(k-m)!\,(k+\lambda)!}{(k-\lambda)!\,(k+m)!}\right]^{1/2} \quad ...(59)$$

$$\times\ \exp\,[i(m-\lambda)\phi](\sin\theta)^{m-\lambda}(1+\cos\theta)^{\lambda}\, P_{k-\lambda}^{m-\lambda,m+\lambda}(\cos\theta)$$

$$\vec{p} = p(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

θ and ϕ vary in the ranges 0 to π and 0 to 2π, respectively, and $P_{-m}^{-\lambda,m+\lambda}(\cos\theta)$ is a Jacobi polynomial.

The expression for $\sigma(\vec{p},\lambda)$ in terms of angular momentum basis is given by (Lomont & Moses 1967) :

$$\sigma(\vec{p},\lambda) = -\left(\frac{8\pi}{3}\right)^{1/2} \sum_{\beta=0,\pm1} (i)^{\beta-\lambda} \vec{\chi}(\beta)\, Y_1^{\beta\lambda *}(\theta,\phi) \quad \text{.. (60)}$$

where,

$$\vec{\chi}(\beta) = (2)^{-1/2}(1, i\beta, 0) \qquad \text{for } \beta = \pm 1$$

$$\vec{\chi}(\beta) = -\, i(0,0,1) \qquad \text{for } \beta = 0.$$

and,

$$Y_1^{\beta\lambda *}(\theta,\phi) = (-1)^{\lambda-\beta}\, Y_1^{\lambda\beta}(\theta,\phi) \quad . \text{ (61)}$$

The expansion for exp $(i\vec{p}.\vec{x})$ in terms of angular momentum basis can be written as :

$$\text{Exp.}\,(i\vec{p}.\vec{x}) = 4\pi \sum_{k=1}^{\infty} \sum_{m=-k}^{k} (i)^k j_k(pr)\, Y_k^{m,0}(\hat{\theta},\hat{\phi})\, Y_k^{m0 *}(\theta,\phi) \quad \text{...(62)}$$

where $r = t \mid x \mid$, j_k is the spherical Bessel's function of order k and θ, $\hat{\phi}$ are the polar angles which describe the direction of x. Now using the equations (58), (60) and (62) in the equation (51a) we get the reductions of electric fields to the inhomogeneous Lorentz group in angular momentum basis :

$$E_1 = 4\sqrt{\pi/3} \sum_{\lambda=\pm1} \sum_{\beta=0,\pm1} \sum_{k=1}^{\infty} \sum_{m=k}^{k} (i)^{k+1+\beta-\lambda}$$

$$\vec{\chi}(\beta) \exp\left[i\pi\left(\lambda - \frac{m}{2}\right)\right]$$

$$\times\, Y_k^{m\lambda}(\theta,\phi)\, Y_1^{\beta\lambda *}(\theta,\phi)\, Y_k^{m0}(\theta,\hat{\phi})\, Y_k^{m0 *}(\theta,\phi)$$

$$\times \int \frac{d\vec{p}}{p}\, j_k(pr) F(p,k,m,\lambda) \exp\,\{ipt\}] \quad \text{...(63)}$$

Equation (63) can further be simplified using Clebsch-Gordan coefficients. Similarly magnetic field may also be expressed as ;

$$H_1(x) = -4\sqrt{\pi/3} \sum_{\lambda=\pm 1} \sum_{\beta=0,\pm 1} \sum_{k=1}^{\infty} \sum_{m=-k}^{k} \lambda \, [(i)^{k+\beta-\lambda} \chi(\vec{\beta}) \exp\left\{i\pi\left(\lambda - \frac{m}{2}\right)\right\}$$

$$\times Y_k^{m\lambda}(\theta,\phi)\, Y_1^{\beta\lambda *}(\theta,\phi)\, Y_k^{mo}(\hat{\theta},\hat{\phi})\, Y_k^{mo *}(\theta,\phi)$$

$$\times \int \frac{dp}{p}\, j_k(pr) F(p,j,m,\lambda) \exp\{-ipt\} \quad .. (64)$$

The plane electromagnetic wave is regular everywhere, including the point $r = 0$. Therefore, only the regular spherical Bessel's function occurs in the expansions (63) and (64).
Hence,

$$j_k(pr) = (pr)^{-1} F_k(r) \quad . (65)$$

$F_k(r)$ in equation (65) is regular function defined as :

$$F_k(r) = \left(\frac{\pi p r}{2}\right)^{1/2} J_{k+1/2}(pr)$$

where $J_{k+1/2}(pr)$ is the regular Bessel's function.

We can also second-quantize the electro-magnetic fields in the angular momentum representation. For this we introduce the annihilation and creation operators, respectively, as :

$$F(p,k,m,\lambda) = a(s)$$

$$F^*(p,k,m,\lambda) = a^*(s)$$

They satisfy the commutation relations :

$$[a(s), a(s')] = [a^*(s), a^*(s')] = 0$$

$$[a(s), a^*(s')] = p\delta(s - s').$$

The annihilation operator in the angular momentum basis is related to annihilation operator in linear momentum basis according to the equation (58). The creation operators are related in complex conjugate manner. The quantized electric field is treated as the sum of operators E_1 and E_1^*, which are obtained by substituting annihilation and creation operator for $F(\vec{p},k,m,\lambda)$ and $F^*(\vec{p},k,m,\lambda)$, respectively, in equation (63) and the similar equation for $E_1^*(x)$. In a similar manner magnetic field is quantized in the angular momentum representation.

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